(Higher) Categorical Galois Theories for the Working Mathematician

Joseph Rennie

Visiting Professor, Colorado College

rrennie@coloradocollege.edu

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How to Crack a Walnut?

- hammer and chisel, or...

> " ...why not simply water? ...through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado! ...the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it. . . yet it finally surrounds the resistant substance. "
>
> — Alexander Grothendieck

Language...

Consider:

- speech and listening

Language...

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- speech and listening
- a vibrating drum (e.g. tympanic membrane)
  - $\rightsquigarrow$ Bessel equations:
  $$x^2y'' + xy' + (x^2 - p^2)y = 0$$

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(Some) other applications:

- wave equations for particles

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(Some) other applications:

- wave equations for particles
- seismology
- structure of DNA

Why stop at understanding vibrations of drums?

Question

To what extent is a differential equation solvable ... in terms of a given set of elementary functions?

E.g.

- Bessel equation solutions are expressible in terms of sin and cos for $p = n + 1/2$, $n \in \mathbb{Z}$

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- otherwise Bessel equation solutions are not expressible in elementary terms
- $y' = 2xy$ has "non-elementary" solutions

Question

Given a

linear ordinary differential equation with solutions, somewhere, (say in $\mathcal{C}^\infty(\mathbb{C})$),

to what extent are its solutions expressible in terms of a given set of elementary functions ?

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Given a

polynomial equation with solutions, *in its algebraic closure, to what extent are its solutions expressible in terms of a given set of elementary numbers?*

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- Galois Theory holds the key

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Why?

Galois Theorem for Finite Field Extensions

Theorem

Fixing a finite galois extension $i : K \hookrightarrow L$, we find

$$\left\{ \begin{array}{c} \textit{intermediate} \\ \textit{splitting extensions} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{subgroups of the} \\ \textit{galois group} \end{array} \right\}$$

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A \arrow[ur, hook] & B \arrow[u, hook] & C \arrow[ul, hook] & & G_1 & G_2 & G_3 \\
 & \cdots & {} \arrow[rr, shift left, "Aut_{(\bullet)}(L)"] & & {} \arrow[ll, shift left, "L^{(\bullet)}"] & \cdots & \\
D & E & F & & G_4 \arrow[dr, hook] & G_5 \arrow[d, hook] & G_6 \arrow[dl, hook] \\
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Can we use this for diff eq's?

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Galois Theorem for Covering Spaces

Theorem

Fixing a regular cover $p : E \twoheadrightarrow B$, *we find*

intermediate normal covering spaces $\simeq$ *subgroups of the fundamental group of B*

$$
\begin{tikzcd}
 & E \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, two heads] & & & & e \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \\
X_1 & X_2 & X_3 & & G_1 & G_2 & G_3 \\
 & \cdots & & {} \arrow[r, shift left, "Aut_{(\bullet)}(E)"] & {} \arrow[l, shift left, "E^{(\bullet)}"] & \cdots & \\
X_4 \arrow[dr, two heads] & X_5 \arrow[d, two heads] & X_6 \arrow[dl, two heads] & & G_4 \arrow[dr, hook] & G_5 \arrow[d, hook] & G_6 \arrow[dl, hook] \\
 & B & & & & Aut_B(E) &
\end{tikzcd}
$$

Galois Theorem for Affine Schemes

Theorem

Fixing a regular cover $p : E \twoheadrightarrow B$*, we find*

intermediate finite étale covers $\simeq$ *subgroups of the fundamental group of B*

$$
\begin{tikzcd}
 & E \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, two heads] & & & & & e \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \\
X_1 & X_2 & X_3 & & & G_1 & G_2 & G_3 \\
 & \cdots & & {} \arrow[r, shift left, "Aut_{(\bullet)}(E)"] & {} \arrow[l, shift left, "E^{(\bullet)}"] & & \cdots & \\
X_4 \arrow[dr, two heads] & X_5 \arrow[d, two heads] & X_6 \arrow[dl, two heads] & & & G_4 \arrow[dr, hook] & G_5 \arrow[d, hook] & G_6 \arrow[dl, hook] \\
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Galois Theorem for Schemes

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 & \cdots \arrow[rrrr, shift left, "Aut_{(\bullet)}(E)"] & & & & \cdots \arrow[llll, shift left, "E^{(\bullet)}"] & \\
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Some Other Instances

- commutative rings (as ‘backward affine schemes)

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Theorem (Borceux et al. 2001)

Given a category $\mathcal{C}$ *with pullbacks, and*

$$F_1, F_2 : \mathcal{C}^{op} \to \underline{\mathrm{Cat}}$$
$$\alpha : F_1 \Rightarrow F_2$$
$$\sigma : L \to K \text{ (in } \mathcal{C}\text{)}$$

such that α *is fully faithful on each component, and* σ *is of effective descent with respect to* F_2*, we have the following equivalence*

$$Split_\alpha(\sigma) \simeq F^{\mathbb{G}_\sigma}$$

Theorem (R. 2022)

Given an ∞-category $\mathcal{C}$ with pullbacks, and

$$F_1, F_2 : \mathcal{C}^{op} \to \mathcal{C}at_\infty$$
$$\alpha : F_1 \Rightarrow F_2$$
$$\sigma : L \to K \ (\text{in } \mathcal{C})$$

such that α is fully faithful on each component, and σ is of effective descent with respect to F_2, we have the following equivalence

$$Split_\alpha(\sigma) \simeq [\![\widehat{\mathbb{G}_\sigma}, F_1]\!]$$

Theorem (Stone Duality)

We have an adjunction:

$$\begin{tikzcd} \mathrm{Bool}^{op} \arrow[r, shift left, "\mathrm{Sp}"] \arrow[r, phantom, "\perp"] & \mathrm{Prof} \arrow[l, shift left, "\mathcal{R}"] \end{tikzcd}$$

Theorem (Stone Duality (restricted))

We have an adjunction:

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Observation

- $\otimes$ *is pullback in* cRing^{op}.
- *For* $R \in \mathrm{cRing}$ *we have* $R\text{-}\mathrm{Alg} \simeq \mathrm{cRing}_{R/}$

$$\mathrm{cRing}^{\mathsf{op}} \underset{\mathcal{R}}{\overset{\mathrm{Sp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Prof}$$

- cRing has pullbacks $\Rightarrow$

$$(\mathrm{cRing}^{\mathsf{op}})_{/S} \underset{\mathcal{R}_s}{\overset{\mathrm{Sp}_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathrm{Prof})_{/\mathrm{Sp}(S)}$$

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$$
\begin{array}{c} \mathcal{C} \\ \big\downarrow F_1 \\ \underline{\mathrm{Cat}} \end{array}
\qquad\qquad
\begin{array}{c} \mathcal{C} \\ \big\downarrow F_2 \\ \underline{\mathrm{Cat}} \end{array}
$$

$$
F_1 \xRightarrow{\ \alpha\ } F_2 \qquad\qquad S \xleftarrow{\ \sigma\ } R
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- The functor $\mathscr{R}_s$ is fully faithful

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- A is split by σ iff $\mathscr{R}(\mathrm{Sp}(S \otimes_R A)) \cong S \otimes_R A$
- If $\sigma : K \to L$ is a galois extension, and $p(x) \in K[x]$ of degree d has r distinct roots, then

$$L^{Hom_{K\text{-}\mathrm{Alg}}(\frac{K[x]}{p(x)}, L)} \cong L^r \cong_{(1)} L^d \cong_{(2)} \frac{L[x]}{p(x)} \cong L \otimes_K \frac{K[x]}{p(x)}$$

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$$\begin{tikzcd} \mathrm{cRing} \arrow[d, "F_1"] & S \arrow[d, mapsto] & \mathrm{cRing} \arrow[d, "F_2"] & S \arrow[d, mapsto] \\ \underline{\mathrm{Cat}} & (\mathrm{Prof})_{/\mathrm{Sp}(S)} & \underline{\mathrm{Cat}} & (\mathrm{cRing}^{\mathsf{op}})_{/S} \end{tikzcd}$$

$$F_1 \overset{\alpha}{\Longrightarrow} F_2 \qquad\qquad S \xleftarrow{\ \sigma\ } R$$

$$\begin{tikzcd} \mathrm{Split}_R(\sigma) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & R\text{-}\mathrm{Alg} \arrow[d, "S\otimes_R \bullet"] \\ (\mathrm{Prof})_{/\mathrm{Sp}(S)} \arrow[r, "\mathscr{R}_s"'] & S\text{-}\mathrm{Alg} \end{tikzcd}$$

Definition

Given a category $\mathcal{C}$, two functors $F_1, F_2 : \mathcal{C}^{\mathsf{op}} \to \underline{\mathrm{Cat}}$, a natural transformation $\alpha : F_1 \Rightarrow F_2$, and an arrow $\sigma : S \to R$ in $\mathcal{C}$, we define $\mathrm{Split}_\alpha(\sigma)$ via the following 2-pullback in $\underline{\mathrm{Cat}}$:

$$\begin{tikzcd} \mathrm{Split}_\alpha(\sigma) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & F_2(R) \arrow[d, "F_2(\sigma)"] \\ F_1(S) \arrow[r, "\alpha_S"'] & F_2(S) \end{tikzcd}$$

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- To get naturality of $\mathcal{R}_s$ from the machinery above requires a 'push-pull formula'

- $\mathrm{Split}_\alpha(\sigma)$ refines $\mathrm{Split}_R(\sigma)$:

$$
\begin{tikzcd}
\mathrm{Split}_R(\sigma)^{\mathsf{op}} \arrow[r, dashed] \arrow[r, phantom, "\lrcorner", very near end, shift left=3] \arrow[d] \arrow[rr, bend left, tail] & \mathrm{Split}_\alpha(\sigma) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R\text{-}\mathrm{Alg}^{\mathsf{op}} \arrow[d, "S \otimes_R \bullet"] \\
\mathrm{Split}_S(1_S) \arrow[r, "\mathrm{Sp}_S"'] & (\mathrm{Prof})_{/\mathrm{Sp}(S)} \arrow[r, tail, "\mathscr{R}_s"'] & S\text{-}\mathrm{Alg}^{\mathsf{op}}
\end{tikzcd}
$$

$$\begin{tikzcd}(\mathrm{cRing}^{\mathsf{op}})_{/S} \arrow[r, shift left, "\mathrm{Sp}_S"] \arrow[r, phantom, "\perp"] & (\mathrm{Prof})_{/\mathrm{Sp}(S)} \arrow[l, shift left, "\mathscr{R}_s"]\end{tikzcd}$$

$$\begin{tikzcd}\mathrm{cRing} \arrow[d, "F_1"] & S \arrow[d, mapsto] & \mathrm{cRing} \arrow[d, "F_2"] & S \arrow[d, mapsto] \\ \underline{\mathrm{Cat}} & (\mathrm{Prof})_{/\mathrm{Sp}(S)} & \underline{\mathrm{Cat}} & (\mathrm{cRing}^{\mathsf{op}})_{/S}\end{tikzcd}$$

$$F_1 \xRightarrow{\ \alpha\ } F_2 \qquad\qquad S \xleftarrow{\ \sigma\ } R$$

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Theorem (Borceux et al. 2001)

Given a category $\mathcal{C}$ with pullbacks, and

$$F_1, F_2 : \mathcal{C}^{op} \to \underline{\mathrm{Cat}}$$
$$\alpha : F_1 \Rightarrow F_2$$
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such that α is fully faithful on each component, and σ is of effective descent with respect to F_2, we have the following equivalence

$$\mathit{Split}_\alpha(\sigma) \simeq F^{\mathbb{G}_\sigma}$$

Theorem (R. 2022)

Given an ∞-category $\mathcal{C}$ with pullbacks, and

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$$\mathit{Split}_\alpha(\sigma) \simeq [\![\widehat{\mathbb{G}_\sigma}, F_1]\!]$$

Some Other Instances

- commutative rings (as ‘backward affine schemes)
- ‘commutative’ ring spectra
  - or sheaves of them $\rightsquigarrow$ differential cohomology
- differential schemes
- logical schemes
- derived schemes
- stable homotopy theories

Thank You

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