

# (Higher) Categorical Galois Theories for the Working Mathematician

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## How to Crack a Walnut?

- hammer and chisel, or...

“ ...why not simply water? ...through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado! ...the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it. . . yet it finally surrounds the resistant substance. ”  
— Alexander Grothendieck

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- speech and listening

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- a vibrating drum (e.g. tympanic membrane)
  - $\rightsquigarrow$  Bessel equations:

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- wave equations for particles
- seismology
- structure of DNA

Why stop at understanding vibrations of drums?



## Question

*To what extent is a differential equation solvable ... in terms of a given set of elementary functions?*

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- otherwise Bessel equation solutions are not expressible in elementary terms
- $y' = 2xy$  has “non-elementary” solutions

## Question

Given a  
*linear ordinary differential equation with solutions, somewhere, (say in  $C^\infty(\mathbb{C})$ ),*

*to what extent are its solutions expressible in terms of a given set of elementary functions      ?*

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- Galois Theory holds the key

Idea: Given a field  $k$  and a polynomial  $p(x) \in k[x]$  the smallest field  $k_p$  containing all of its roots. (A.K.A. "splitting field" of  $p$  over  $k$ )

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## Why?

# Galois Theorem for Finite Field Extensions

## Theorem

Fixing a finite galois extension  $i: K \hookrightarrow L$ , we find

*intermediate*

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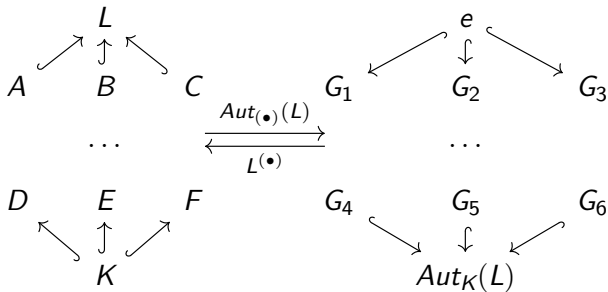
*splitting extensions*

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Can we use this for diff eq's?

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# STOP!!!!



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# Galois Theorem for Finite Field Extensions

## Theorem

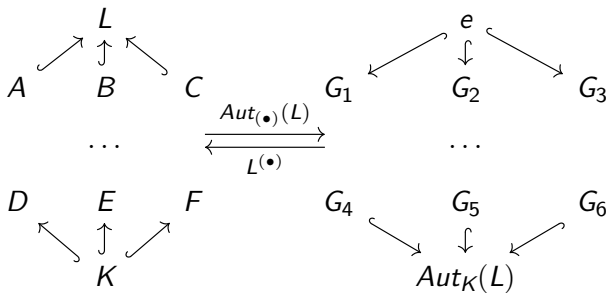
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# Galois Theorem for Covering Spaces

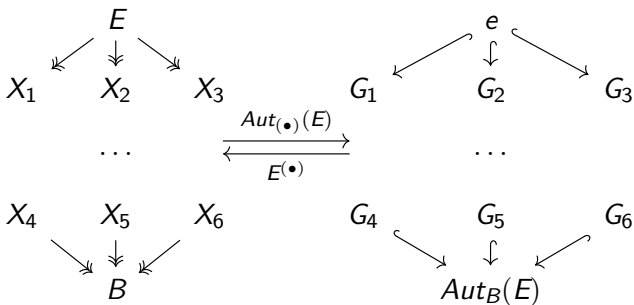
## Theorem

Fixing a *regular cover*  $p : E \rightarrow B$ , we find

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# Galois Theorem for Affine Schemes

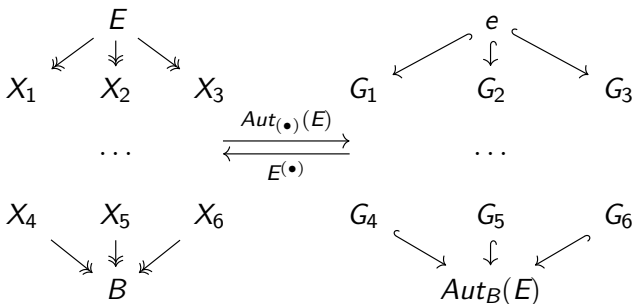
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# Galois Theorem for Schemes

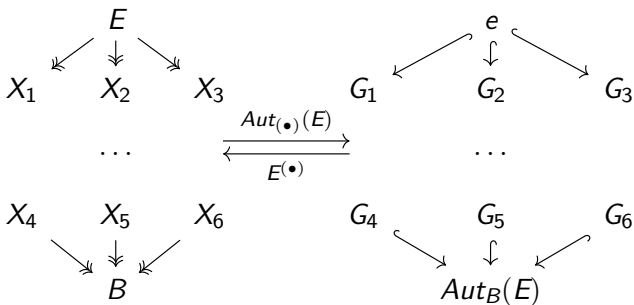
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# Some Other Instances

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## Theorem (Borceux et al. 2001)

Given a category  $\mathcal{C}$  with pullbacks, and

$$F_1, F_2 : \mathcal{C}^{op} \rightarrow \underline{\text{Cat}}$$

$$\alpha : F_1 \Rightarrow F_2$$

$$\sigma : L \rightarrow K \text{ (in } \mathcal{C} \text{)}$$

such that  $\alpha$  is fully faithful on each component, and  $\sigma$  is of effective descent with respect to  $F_2$ , we have the following equivalence

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## Theorem (R. 2022)

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## Theorem (Stone Duality)

*We have an adjunction:*

$$\text{Bool}^{\text{op}} \begin{array}{c} \xrightarrow{\text{Sp}} \\ \perp \\ \xleftarrow{\mathcal{R}} \end{array} \text{Prof}$$

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## Observation

- $\otimes$  is pullback in  $\text{cRing}^{op}$ .
- For  $R \in \text{cRing}$  we have  $R\text{-Alg} \simeq \text{cRing}_{R/}$

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- If  $\sigma : K \rightarrow L$  is a galois extension, and  $p(x) \in K[x]$  of degree  $d$  has  $r$  distinct roots, then

$$L^{\text{Hom}_{K\text{-Alg}}(\frac{K[x]}{p(x)}, L)} \cong L^r \cong_{(1)} L^d \cong_{(2)} \frac{L[x]}{p(x)} \cong L \otimes_K \frac{K[x]}{p(x)}$$

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## Definition

Given a category  $\mathcal{C}$ , two functors  $F_1, F_2 : \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Cat}}$ , a natural transformation  $\alpha : F_1 \Rightarrow F_2$ , and an arrow  $\sigma : S \rightarrow R$  in  $\mathcal{C}$ , we define  $\text{Split}_\alpha(\sigma)$  via the following 2-pullback in  $\underline{\text{Cat}}$ :

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 \end{array}$$

- To get naturality of  $\mathcal{R}_S$  from the machinery above requires a 'push-pull formula'

- $\text{Split}_\alpha(\sigma)$  refines  $\text{Split}_R(\sigma)$ :

$$\begin{array}{ccccc}
 \text{Split}_R(\sigma)^{\text{op}} & & \text{Split}_\alpha(\sigma) & \xrightarrow{\quad} & R\text{-Alg}^{\text{op}} \\
 \downarrow & \dashrightarrow & \downarrow & & \downarrow S \otimes_R \bullet \\
 \text{Split}_S(1_S) & \xrightarrow{\text{Sp}_S} & (\text{Prof})_{/\text{Sp}(S)} & \xrightarrow{\mathcal{R}_S} & S\text{-Alg}^{\text{op}}
 \end{array}$$

A curved arrow labeled  $\text{'}\text{'}$  points from  $\text{Split}_R(\sigma)^{\text{op}}$  to  $\text{Split}_\alpha(\sigma)$ .

$$(\text{cRing}^{\text{op}})_{/S} \begin{array}{c} \xrightarrow{\text{Sp}_S} \\ \xleftarrow[\mathcal{R}_S]{\perp} \end{array} (\text{Prof})_{/Sp(S)}$$

$$\begin{array}{ccc} \text{cRing} & S & \text{cRing} & S \\ \downarrow F_1 & \downarrow & \downarrow F_2 & \downarrow \\ \underline{\text{Cat}} & (\text{Prof})_{/Sp(S)} & \underline{\text{Cat}} & (\text{cRing}^{\text{op}})_{/S} \end{array}$$

$$F_1 \xrightarrow{\alpha} F_2 \qquad S \xleftarrow{\sigma} R$$

$$\begin{array}{ccc} \text{Split}_{\alpha}(\sigma) & \dashrightarrow & F_2(R) \\ \downarrow & \lrcorner & \downarrow F_2(\sigma) \\ F_1(S) & \xrightarrow{\alpha_S} & F_2(S) \end{array}$$

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## Theorem (Borceux et al. 2001)

Given a category  $\mathcal{C}$  with pullbacks, and

$$F_1, F_2 : \mathcal{C}^{op} \rightarrow \underline{\text{Cat}}$$

$$\alpha : F_1 \Rightarrow F_2$$

$$\sigma : L \rightarrow K \text{ (in } \mathcal{C}\text{)}$$

such that  $\alpha$  is fully faithful on each component, and  $\sigma$  is of effective descent with respect to  $F_2$ , we have the following equivalence

$$\text{Split}_\alpha(\sigma) \simeq F^{\text{G}\sigma}$$

## Theorem (R. 2022)

Given an  $\infty$ -category  $\mathcal{C}$  with pullbacks, and

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such that  $\alpha$  is fully faithful on each component, and  $\sigma$  is of effective descent with respect to  $F_2$ , we have the following equivalence

$$\mathit{Split}_\alpha(\sigma) \simeq \llbracket \widehat{G}_\sigma, F_1 \rrbracket$$

## Some Other Instances

- commutative rings (as 'backward affine schemes')
- 'commutative' ring spectra
  - or sheaves of them  $\rightsquigarrow$  differential cohomology
- differential schemes
- logical schemes
- derived schemes
- stable homotopy theories

*Thank You*

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