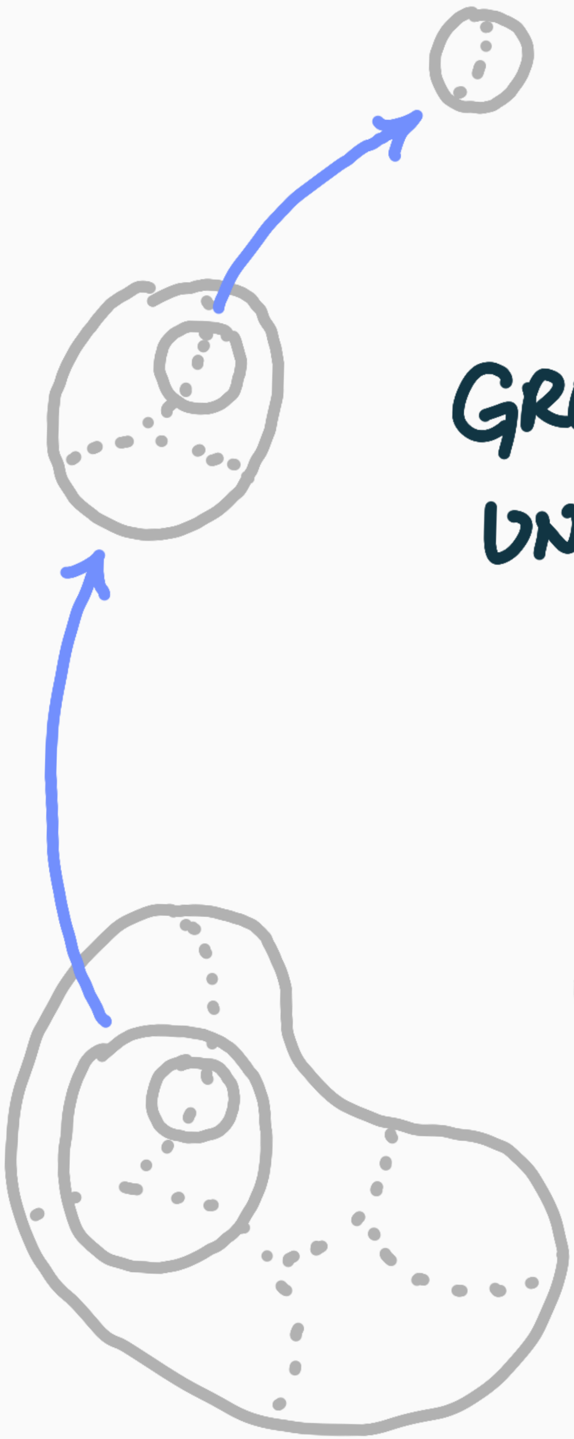

ENRICHED GROTHENDIECK TOPOLOGIES UNDER CHANGE OF BASE

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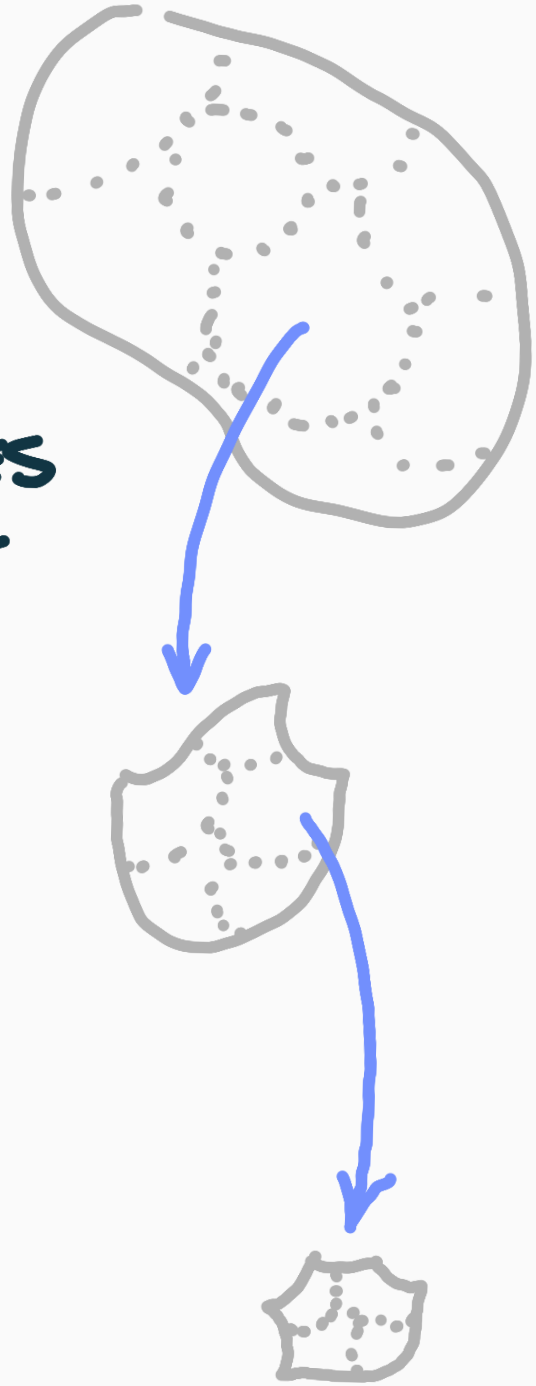


Background.

BACKGROUND

- CLASSICAL GEL'FAND DUALITY: AN EQUIVALENCE OF CATEGORIES

$$\text{cpt Haus} \underset{\mathrm{Spec}}{\overset{\mathcal{C}_0(-)}{\rightleftarrows}} \text{comm } C^*\mathrm{Alg}^{op}$$

STATE SPACES

OBSERVABLES ON CLASSICAL SYSTEMS

NONCOMMUTATIVE SPECTRAL THEORY

WHAT IF I WANT GEL'FAND DUALITY

FOR NON COMMUTATIVE C^*-ALGEBRAS?

(i.e., OBSERVABLES ON QUANTUM SYSTEMS?)

THAT WOULD BE SO COOL...

THEN I COULD DO

ACTUAL GEOMETRY...

I COULD TRY JUST APPLYING THE Spec

FUNCTOR I ALREADY HAVE??

NONCOMMUTATIVE SPECTRAL THEORY

PROBLEM! (REYES, 2012 ; v. d. BERGH – HEUNEN, 2014)

THEOREM: ANY FUNCTOR $F : C^*Alg^{op} \longrightarrow Top$

FOR WHICH

$$\begin{tikzcd} \mathrm{comm}\, C^*Alg^{op} \arrow[r, "Spec"] \arrow[d, hook, "inc"'] & Top \\ C^*Alg^{op} \arrow[ur, "F"'] & \end{tikzcd}$$

COMMUTES MUST HAVE $F(M_n(\mathbb{C})) \cong \emptyset$

FOR $n \geq 3$.

NONCOMMUTATIVE SPECTRAL THEORY

IN PHYSICAL TERMS: (I'LL DO MY BEST!)

- ANY CLASSICAL SYSTEM (comm. C^*-Alg) IS DETERMINED BY A STATE SPACE (CPT. TOP. SPACE) IN A WAY THAT RESPECTS OPERATIONS ON THE SYSTEM (FUNCTORIALLY).

- NO COMMUTATIVE C^*-ALG DETERMINES A QUANTUM SYSTEM IN THIS WAY.

NONCOMMUTATIVE SPECTRAL THEORY

- THE DREAM OF NONCOMMUTATIVE GEOMETRY:

$$\begin{tikzcd}
\mathrm{comm}\,C^*\mathrm{Alg}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{cpt\,Haus} \arrow[d, hook, dashed, "??"] \\
C^*\mathrm{Alg}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``NONCOMMUTATIVE CPT. HAUSDORFF SPACES''}
\end{tikzcd}$$

OBSERVABLES ON QUANTUM SYSTEMS: THESE ARE WHAT N.C. GEOMETRY STUDIES.

NONCOMMUTATIVE SPECTRAL THEORY

- TOPOLOGICAL SPACES ARE SETS OF POINTS, SO I COULD JUST AS WELL THINK ABOUT A SLIGHTLY SIMPLER PROBLEM:

$$
\begin{tikzcd}
\mathrm{commRing}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Set} \arrow[d, hook, dashed, "??"] \\
\mathrm{Ring}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``}q\mathrm{Set}\text{''}
\end{tikzcd}
$$

FINDING A CAT. OF "QUANTUM / N.C." SETS.

NONCOMMUTATIVE SPECTRAL THEORY

WE EXPECT "qSet" TO BE A CLOSED SYMMETRIC MONOIDAL CATEGORY, AND WE EXPECT IT TO "CONTAIN" Set.

HOW TO TELL IF A GIVEN CANDIDATE, SAY $\mathcal{V} \in \mathrm{MonCat}$, IS THE RIGHT ONE?

ENRICHMENT

- THINK OF AN ENRICHED CATEGORY AS A CATEGORY WHOSE HOM SETS HAVE EXTRA STRUCTURE.

  ENRICHED FUNCTORS, TRANSFORMATIONS RESPECT THIS EXTRA STRUCTURE.

  e.g., A RING IS A ONE-OBJECT Ab-CATEGORY. A RING HOMOMORPHISM IS AN Ab-FUNCTOR.

ENRICHMENT

- GIVEN ENRICHING CAT'S $\mathcal{U}$, $\mathcal{V}$, A $\mathcal{V}$-CAT. $\mathcal{C}$, AND A FUNCTOR
$$G : \mathcal{V} \longrightarrow \mathcal{U},$$
WE CAN CHANGE THE BASE OF $\mathcal{C}$ VIA G, OBTAINING A $\mathcal{U}$-CAT.

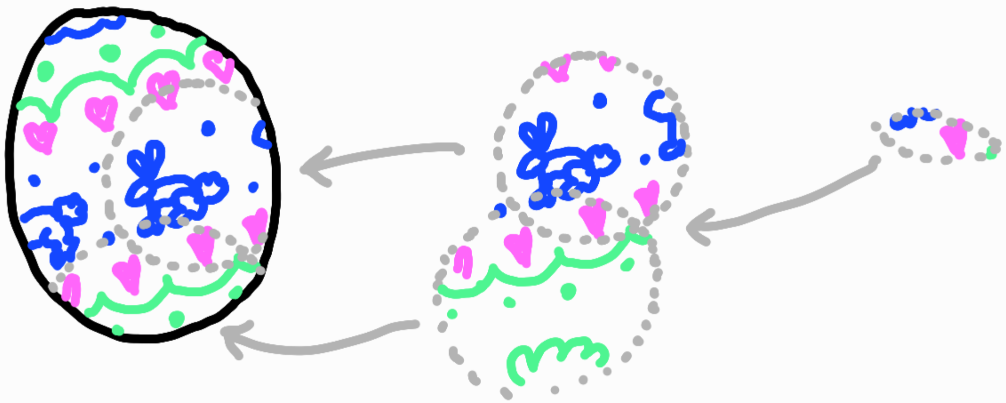
e.g., USING $\mathrm{Hom}_{Ab}(\mathbb{Z}, -) : Ab \longrightarrow Set$, WE CAN "FORGET" THE ADDITIVE STRUCTURE ON A RING TO OBTAIN A MERE MULTIPLICATIVE MONOID.

SHEAVES

- A GROTHENDIECK TOPOLOGY IS A WAY TO ENDOW ANY CATEGORY (NOT JUST $\mathcal{O}(X)$) WITH A NOTION OF "CLOSE-TOGETHER-NESS" OF ITS OBJECTS.

SHEAVES

- WE WANT TO KNOW WHEN OBJECTS ARE CLOSE TOGETHER BECAUSE THEN WE CAN DEFINE SHEAVES: "CLOSENESS-RESPECTING" ASSIGNMENTS OF DATA TO THE "LOCATIONS" IN (i.e., OBJECTS OF) OUR CAT'S.



ENRICHED SHEAVES!

WE HAVE ENRICHED GROTHENDIECK TOPOLOGIES AND ENRICHED SHEAVES, TOO (SAY, OVER $\mathcal{V}$).

e.g., A GABRIEL LOCALIZING SYSTEM ON A RING IS AN Ab-TOPOLOGY.
AN Ab-SHEAF ON A RING W/R/T SUCH A LOCALIZING SYSTEM IS ITS GABRIEL LOCALIZATION.

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

ONE WAY TO TEST WHETHER WE'VE FOUND A GOOD CANDIDATE FOR "qSet" — THUS A GOOD CAT. TO MODEL QUANTUM SYSTEMS WITH — IS TO MAKE SURE THERE ARE qSet-ENRICHED GROTH. TOPOLOGIES WHICH DON'T ARISE FROM UNENRICHED ONES.

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

SO, NEED TOOLS TO DETECT WHETHER A GIVEN ENRICHED GROTHENDIECK TOPOLOGY ARISES FROM AN UNENRICHED ONE.

WHAT I WANTED TO KNOW:

1. HOW CAN I USE BASE CHANGE TO TURN A $\mathcal{V}$-TOPOLOGY INTO A $\mathcal{U}$-TOPOLOGY?

2. UNDER WHAT CONDITIONS IS SUCH AN ASSIGNMENT INJECTIVE?

Some Key Results.

WHAT IS A GROTHENDIECK TOPOLOGY?

BORCEUX - QUINTEIRO, 1996 :

Definition 1.2 *Let $\mathcal{C}$ be a small* V- *category. A* V- *Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable* V- *presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ *for every object* $C \in \mathcal{C}$;

(T2) given $R \in \mathcal{T}(C)$ *and* $f \in_G \mathcal{C}(D,C)$, *one has* $f^{-1}(R) \in \mathcal{T}(D)$, *where* $f^{-1}(R)$ *is defined by the following pullback:*

$$\begin{tikzcd}
f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G,R\} \arrow[d, tail] \\
\mathcal{C}(-,D) \arrow[r, "f"'] & \{G,\mathcal{C}(-,C)\};
\end{tikzcd}$$

(T3) given $S \in \mathcal{T}(C)$ *and a subobject* $R \rightarrowtail \mathcal{C}(-,C)$ *such that* $f^{-1}(R) \in \mathcal{T}(D)$ *for all* $f \in_G S(D)$, *one has* $R \in \mathcal{T}(C)$.

WHAT IS A GROTHENDIECK TOPOLOGY?

COVERINGS

IDEALS

FUNCTORS

"A FAMILY OF ______

SATISFYING SOME CLOSURE

CONDITIONS."

COVERAGE: ASSIGNMENT SATISFYING ONLY (T1) AND (T2).

DENOTE

$$\Sigma(\mathcal{C}, \mathcal{V}) = \{\text{ALL } \mathcal{V}\text{-COVERAGES ON } \mathcal{C}\}$$

$$\tau(\mathcal{C}, \mathcal{V}) = \{\text{ALL } \mathcal{V}\text{-TOPOLOGIES ON } \mathcal{C}\}$$

ENRICHING CATEGORIES $\mathcal{U}, \mathcal{V}$ ARE ASSUMED TO BE:

- CLOSED SYMMETRIC MONOIDAL
- LOCALLY FINITELY PRESENTABLE
- BARR REGULAR
- BICOMPLETE "W/R/T Set" ($\mathcal{U}_0, \mathcal{V}_0$ BICOMP.)

WE CONSIDER LAX MONOIDAL FUNCTORS

$G : \mathcal{V} \to \mathcal{U}$ WHICH MAY BE

$$\left.\begin{array}{l} \text{i. FAITHFUL} \\ \text{ii. CONSERVATIVE} \\ \text{iii. RIGHT ADJOINTS IN MonCat} \end{array}\right\} (\blacktriangle)$$

BASE CHANGE

FIX $\mathcal{U}, \mathcal{V}$, AND $G : \mathcal{V} \to \mathcal{U}$.

G DETERMINES A 2-FUNCTOR

$$\mathcal{V}\text{-}Cat \xrightarrow{G_*} \mathcal{U}\text{-}Cat .$$

GIVEN A $\mathcal{V}$-CATEGORY $\mathcal{C}$, DENOTE THE CORRESPONDING $\mathcal{U}$-CATEGORY BY $G_*\mathcal{C}$.

BASE CHANGE

IF $R : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ IS A $\mathcal{V}$-PRESHEAF, WE OBTAIN A $\mathcal{U}$-FUNCTOR $G_* R : G_* \mathcal{C}^{op} \longrightarrow G_* \mathcal{V}$

— NOT QUITE A $\mathcal{U}$-PRESHEAF!

IT DOESN'T TAKE VALUES IN $\mathcal{U}$.

BASE CHANGE

MONOIDAL
IN CASE G IS A RIGHT ADJOINT,

WE HAVE AN INDUCED ADJUNCTION

$$G_* \mathcal{V} \underset{}{\overset{}{\rightleftarrows}} \mathcal{U} \quad \top$$

IN $\mathcal{U}\text{-}\mathcal{C}at$.

BASE CHANGE

WE CAN USE A COMPONENT OF THIS INDUCED $\mathcal{U}$-ADJUNCTION TO TURN $G_* R$ INTO A $\mathcal{U}$-PRESHEAF

$$\widetilde{G}R : G_* \mathcal{C}^{op} \longrightarrow \mathcal{U} .$$

BASE CHANGE

PROPOSITION. SUPPOSE G IS FAITHFUL AND A RIGHT ADJOINT. GIVEN A $\mathcal{V}$-SUBFUNCTOR $R \rightarrowtail \mathcal{C}(-,x)$, G INDUCES A $\mathcal{U}$-SUBFUNCTOR

$$\tilde{G}R \rightarrowtail \tilde{G}\mathcal{C}(-,x).$$

SUPPOSE $\mathcal{C}$ IS SMALL AND G IS A FAITHFUL RIGHT ADJOINT.

<u>PROPOSITION</u>. GIVEN A $\mathcal{V}$-COVERAGE J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-COVERAGE $\tilde{G}J$ ON $G_*\mathcal{C}$.

PROPOSITION. GIVEN A $\mathcal{V}$-TOPOLOGY J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-TOPOLOGY $\overline{GJ}$ ON $G_*\mathcal{C}$.

IDEA: FORM $\tilde{G}J$, THEN "CLOSE UP" USING A TRANSFINITE CONSTRUCTION.

THEOREM. $\Sigma(\mathscr{C}, \mathcal{V})$ AND $\mathcal{T}(\mathscr{C}, \mathcal{V})$ ARE COMPLETE LATTICES.

SAME FOR $G_*\mathscr{C}$, $\mathcal{U}$, ETC.

THEOREM. SUPPOSE G SATISFIES ALL CONDITIONS IN ($\blacktriangle$). THE ASSIGNMENT

$$\Sigma(\mathcal{C}, \mathcal{V}) \xrightarrow{\tilde{G}(-)} \Sigma(G_* \mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

CONJECTURE. SUPPOSE G SATISFIES ALL CONDITIONS IN ($\blacktriangle$), PLUS ONE MORE (SECRET) CONDITION. THE ASSIGNMENT

$$\mathcal{T}(\mathcal{C},\mathcal{V}) \xrightarrow{\overline{G(-)}} \mathcal{T}(G_*\mathcal{C},\mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

THIS WOULD LET US MAKE USE OF BIJECTIVE CORRESPONDENCES (BORCEUX - QUINTEIRO, 1994)

$$\mathcal{T}(\mathcal{C}, \mathcal{V}) \longleftrightarrow \left\{ \begin{array}{c} \text{LOCALIZATIONS OF} \\ [\mathcal{C}^{op}, \mathcal{V}] \end{array} \right\}$$

"ENRICHED SHEAF TOPOSES"

$$\longleftrightarrow \left\{ \begin{array}{c} \text{UNIVERSAL CLOSURE} \\ \text{OPERATIONS ON} \\ [\mathcal{C}^{op}, \mathcal{V}] \end{array} \right\}.$$

PROBLEM: $\overline{G(-)}$ IS HARD TO WORK WITH - NEED AN "EASIER" WAY TO CONSTRUCT.

IT APPEARS THAT FAITHFULNESS IS NECESSARY!

THEOREM. FOR A FIELD k, $\mathcal{V} = \mathrm{grMod}_k$, $\mathcal{U} = \mathrm{Set}$, AND

NOT FAITHFUL!

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathcal{U},$$

THERE EXIST DISTINCT $\mathcal{V}$-COVERAGES ON $k[x, y]$ WHICH GIVE RISE TO THE SAME $\mathcal{U}$-COVERAGE.

ONE MORE THING

THEOREM. IN CASE $G : \mathcal{V} \to \mathcal{U}$ IS FULLY FAITHFUL, BASE CHANGE "COMMUTES" WITH ENRICHED SHEAFIFICATION.

UP NEXT

- (J/W ANA LUIZA TENÓRIO) PROVE AN ENRICHED VERSION OF STREET'S CLASSIFICATION THEOREM:

  STREET, 1981: GROTH. TOPOSES ARE EXACTLY THOSE CATEGORIES WHOSE YONEDA EMBEDDING HAS A LEFT-EXACT LEFT ADJOINT.

UP NEXT

- THE $\mathcal{V}$-SHEAVES ON A $\mathcal{V}$-CATEGORY THEMSELVES FORM A $\mathcal{V}$-CATEGORY, WHICH IS ONE WAY TO SAY WHAT A $\mathcal{V}$-TOPOS IS. CAN I CHARACTERIZE $\mathcal{V}$-TOPOSES AXIOMATICALLY? (i.e., PROVE AN ENRICHED VERSION OF GIRAUD'S THEOREM?)

THANKS FOR LISTENING!

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